Int. J. Solids Structures Vol. 26, No. 5/6, pp. 663–674, 1990
Printed in Great Britain.

0020-7683/90 $3.00+.00

THE MECHANICAL PROPERTIES OF PLATELET REINFORCED COMPOSITES

A. N. NORRIS
Department of Mechanics and Materials Science, Rutgers University, Piscataway, NJ 08855-0909, U.S.A.

(*Received* 11 *April* 1989; *in revised form* 28 *July* 1989)

Abstract—The effective moduli of platelet reinforced media are derived for aligned and randomly oriented circular platelets at both dilute and finite concentrations. The platelets are modeled as very thin oblate spheroids in which edge effects caused by the presence of the sharp corners can be significant, depending upon the relative magnitudes of the thickness-to-diameter ratio and the ratio of the matrix stiffness to that of the reinforcer. The edge effects become negligible when the latter ratio greatly exceeds the former, in which case the platelets act effectively as infinite layers. In general, the non-uniform stress fields in the vicinity of the sharp corners or edges cause a reduction in the effective moduli. When the aspect ratio greatly exceeds the stiffness ratio, the inclusions become equivalent to rigid disks, and the pertinent concentration parameter is not the volume fraction, which is zero, but a number analogous to the crack density parameter for solids containing cracks. Effective medium theories for finite concentrations of rigid disks predict that the effective Poisson's ratio tends to the value 0.1557... as the concentration increases, and the self-consistent theory displays a critical disk density at which the composite becomes rigid.

1. INTRODUCTION

The subject of this paper is the stiffening effect of platelet reinforcers in two-phase composite materials. While aligned fibers are known to be very efficient for providing uniaxial stiffening, randomly oriented short fibers do not compare with platelets for purposes of stiffening in all directions simultaneously. In fact, there is considerable evidence that the maximum isotropic stiffening for a given volume of filler can be achieved using thin platelets. Experimental data, summarized by Maine and Shepherd (1974), shows that the Young's modulus of a plastic matrix composite is significantly improved when mica flakes are used, as compared with glass or boron fibers. The same general conclusions about the efficacy of platelets versus other filler shapes are predicted by the theoretical investigations of, for instance, Wu (1966), Walpole (1969), Boucher (1974) and Christensen (1979a, b).

Previous theoretical studies (Wu, 1966; Walpole, 1969; Boucher, 1974; Christensen, 1979a, b) have assumed that the effects due to the non-uniform stress fields in the vicinity of the sharp corners of the platelet are insignificant, and consequently the inclusion may be viewed as a layer of infinite lateral extent, an approximation which leads to considerable simplification. In particular, it can be shown that the dilute concentration predictions for randomly oriented layers coincide with the Hashin–Shtrikman (Christensen, 1979a; Hashin and Shtrikman, 1962) upper bounds for the bulk and shear moduli of the composite (Boucher, 1974; Christensen 1979b). The dilute concentration results may be extended to finite concentrations by using effective medium approximations. Thus, Wu (1966), Walpole (1969) and Boucher (1974) each employed the self-consistent technique of Hill (1965) and Budiansky (1965), but only Boucher recognized that the predicted moduli are the same as the bounds. Christensen (1979a, b) presented a theory that predicts moduli in accord with the bounds for finite concentrations, and Norris (1985) has noted that the Mori–Tanaka effective medium approximation yields the upper bounds.

In practice, the maximum stiffness corresponding to the bounding moduli cannot be achieved because the platelets are not infinite in extent. The stress concentrations in the matrix near the sharp corners of the platelets cause a reduction in stiffness. Whether or not this effect is important depends, as we will see, upon the relative magnitude of the platelet aspect ratio and the ratio of material stiffnesses in the matrix and filler. The corner or edge effects are included in the present theory by modeling the platelet as a very thin oblate spheroid. Previous theoretical results are obtained in the limit of vanishing thickness.

However, for finite values of the aspect ratio, the edge effects are crucial for considering the limit of a rigid filler. Then the platelet acts as the dual to a crack, in which the compliance is infinite. As we will see, there is considerable similarity between the present theory for rigid platelets and the theory of solids containing aligned (Laws and Dvorak, 1987) and randomly oriented (Budiansky and O'Connell, 1976; Zimmerman, 1985; Hashin, 1988) penny-shaped cracks.

After mathematical preliminaries in Section 2, the fundamental result for a dilute concentration of aligned platelets is derived in Section 3. Dilute concentrations of randomly oriented platelets are considered in Section 4. Lower bounds on the moduli for finite concentrations are established in Section 5. Finally, in Section 6, the special case of rigid platelets is considered at length, and several effective medium theories are discussed.

2. NOTATION AND PRELIMINARY RESULTS

Let **L** and **M** denote tensors of stiffness and compliance, respectively, with components L_{ijkl} and M_{ijkl} referred to a rectangular coordinate system, and $i, j, k, l = 1, 2, 3$. Both modulus tensors satisfy the symmetries

$$A_{ijkl} = A_{klij} = A_{jikl}. \tag{1}$$

Tensor products are defined by

$$(AB)_{ijkl} = A_{ijmn}B_{mnkl}, \quad \text{(sum over } m, n) \tag{2}$$

and so

$$\mathbf{LM} = \mathbf{ML} = \mathbf{I}$$

where **I** is the identity, $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$. Isotropic tensors of the form

$$A_{ijkl} = \frac{\alpha}{3}\delta_{ij}\delta_{kl}+\beta\left(I_{ijkl}-\frac{1}{3}\delta_{ij}\delta_{kl}\right) \tag{3}$$

will be denoted concisely as (α, β); thus, products are $(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\beta_2)$, and $\mathbf{I} = (1, 1)$.

We use the concise notation of Walpole (1966a) for transversely isotropic fourth order tensors. For example, a transversely isotropic solid with x_3 as the symmetry axis possesses five independent stiffness moduli, C_{11}, C_{13}, C_{33}, C_{44} and C_{66}, and in Walpole's notation

$$\mathbf{L} = (2C_{11}-2C_{66},\ C_{13},\ C_{13},\ C_{33},\ 2C_{66}, 2C_{44}).$$

In general, a transversely isotropic tensor **A** with the symmetries $A_{ijkl} = A_{jikl} = A_{ijlk}$ is characterized by six constants, and may be denoted

$$\mathbf{A} = (2\alpha,\ \beta_1,\ \beta_2,\ \gamma,\ 2\delta,\ 2\varepsilon). \tag{4}$$

The identity is $\mathbf{I} = (1, 0, 0, 1, 1, 1)$, the inverse of **A** is

$$\mathbf{A}^{-1} = \frac{1}{2}\left(\frac{\gamma}{\alpha\gamma-\beta_1\beta_2}, \frac{-\beta_1}{\alpha\gamma-\beta_1\beta_2}, \frac{-\beta_2}{\alpha\gamma-\beta_1\beta_2}, \frac{2\alpha}{\alpha\gamma-\beta_1\beta_2}, \frac{1}{\delta}, \frac{1}{\varepsilon}\right) \tag{5}$$

and the product of **A** with $\mathbf{B} = (2a, b_1, b_2, c, 2d, 2e)$ is

$$\mathbf{AB} = (4\alpha a+2\beta_1 b_2,\ 2\alpha b_1+\beta_1 c,\ 2\beta_2 a+\gamma b_2,\ 2\beta_2 b_1+\gamma c,\ 4\delta d,\ 4\varepsilon e). \tag{6}$$

3. A DILUTE CONCENTRATION OF ALIGNED PLATELETS

Consider a matrix of moduli $\mathbf{L}_m$ containing a dilute concentration of aligned, identical inclusions of material with moduli $\mathbf{L}_i$. The effective moduli by definition relate the average stress to the average strain in the composite. Let $\boldsymbol{\varepsilon}$ be the average strain and suppose the average strain in the inclusions is $\boldsymbol{\varepsilon}_i = \mathbf{T}\boldsymbol{\varepsilon}$, where $\mathbf{T}$ is the fourth order strain concentration tensor, then it can be easily shown that the effective moduli are

$$\mathbf{L} = \mathbf{L}_m + c(\mathbf{L}_i - \mathbf{L}_m)\mathbf{T} + 0(c^2). \tag{7}$$

where $c \ll 1$ is the volume fraction of inclusions.

If the inclusion shape is ellipsoidal we can avail of Eshelby's fundamental result for an isolated inclusion that

$$\mathbf{T} = [(\mathbf{I} + \mathbf{P}(\mathbf{L}_i - \mathbf{L}_m)]^{-1} \tag{8}$$

where $\mathbf{P}$ satisfies the symmetries (1) and is related to Eshelby's $\mathbf{S}$ tensor by $\mathbf{P} = \mathbf{S}\mathbf{M}_m$, see Hill (1965) for further discussion of these tensorial identities. The tensor $\mathbf{P}$ depends only upon the matrix moduli and the aspect ratios of the inclusion.

Let the matrix be isotropic, with bulk modulus κ_m, shear modulus μ_m and Poisson's ratio $v_m = (3\kappa_m - 2\mu_m)/(6\kappa_m + 2\mu_m)$, so that $\mathbf{L}_m = (3\kappa_m, 2\mu_m)$. The inclusion is assumed to be a thin, oblate spheroid of radius a and aspect ratio $\alpha \ll 1$, which is aligned with the normal to its face in the x_3 direction. The associated limiting form of S, and hence $\mathbf{P}$, follows from Mura (1982) as

$$\mathbf{P} = \frac{1}{2\mu_m}\left(0, 0, 0, \frac{1-2v_m}{1-v_m}, 0, 1\right) + \frac{\pi\alpha}{16\mu_m(1-v_m)}\left(3-4v_m, -1, -1, -2(1-4v_m), \frac{7}{2}-4v_m, -2(2-v_m)\right) + 0(\alpha^2). \tag{9}$$

The platelet is also composed of isotropic material, with moduli κ_i, μ_i and Poisson's ratio v_i. By assumption, the material in the platelet is much stiffer than the matrix, i.e., $\kappa_i \gg \kappa_m$ and $\mu_i \gg \mu_m$. Then, from eqns (7) and (8),

$$\mathbf{L} \approx \mathbf{L}_m + c(\mathbf{P} + \mathbf{M}_i)^{-1} \tag{10}$$

where

$$\mathbf{M}_i = \frac{1}{2\mu_i(1+v_i)}(1-v_i, -v_i, -v_i, 1, 1+v_i, 1+v_i) \tag{11}$$

and the volume fraction of platelets is

$$c = \tfrac{4}{3}\pi a^3 \alpha n \tag{12}$$

n being the number of platelets per unit volume.

The change in moduli due to the presence of the platelets depends upon the tensor $(\mathbf{P} + \mathbf{M}_i)^{-1}$, in which both $\mathbf{P}$ and $\mathbf{M}_i$ involve small quantitites, i.e., α in $\mathbf{P}$, and the ratio μ_m/μ_i for $\mathbf{M}_i$. To proceed further, we assume some scaling between these quantities. The limiting cases of ultimate interest will be:

(i) $\alpha \ll \mu_m/\mu_i$; and (ii) $\alpha \gg \mu_m/\mu_i$. Case (i) corresponds to the limit of a "large" platelet in which effects due to the presence of the sharp corners or edges of the platelet are unimportant, and case (ii) is realized when the inclusion is so stiff that only these effects matter, i.e., the platelet becomes in effect a rigid disk. As we will see, both limits may be considered by first assuming

$$\alpha = 0(\mu_m/\mu_i). \tag{13}$$

The change in moduli then follows from eqns (5) and (9)–(13) as

$$\mathbf{L}-\mathbf{L}_m = \left(\left[\frac{3(3-4\nu_m)}{\varepsilon 64\mu_m(1-\nu_m)}+\frac{1}{c2\mu_i}\left(\frac{1-\nu_i}{1+\nu_i}\right)\right]^{-1}, 0, 0, 0, \left[\frac{3(7-8\nu_m)}{\varepsilon 128\mu_m(1-\nu_m)}+\frac{1}{c2\mu_i}\right]^{-1}, 0\right)+0(c\mathbf{L}_m) \tag{14}$$

where ε is the disk density parameter,

$$\varepsilon = na^3. \tag{15}$$

Equation (14) is the fundamental result of this paper. The first thing to note is that the platelets only alter the in-plane moduli, i.e., C_{11} and C_{66} if the platelets are aligned perpendicular to the x_3-axis. In the limit as the aspect ratio α tends to zero, the terms in (14) involving c dominate, to give

$$\mathbf{L}-\mathbf{L}_m = c2\mu_i\left(\frac{1+\nu_i}{1-\nu_i}, 0, 0, 0, 1, 0\right). \tag{16}$$

This is precisely the same as the change in moduli caused by a dilute concentration of parallel infinite layers of inclusion material (Christensen, 1979a, b; Postma, 1955). Conversely, when the included material is extremely stiff, tending to the rigid limit $\mu_i \to \infty$, the terms in (14) with ε dominate, and

$$\mathbf{L}-\mathbf{L}_m = \varepsilon\frac{64}{3}\mu_m(1-\nu_m)\left(\frac{1}{3-4\nu_m}, 0, 0, 0, \frac{2}{7-8\nu_m}, 0\right). \tag{17}$$

Willis (1980, 1981) gave a similar expression for a composite containing flat rigid platelets; however, there appears to be a typographical error in his expressions [1980, eqn (5.25); 1981, eqn (4.101)].

The crossover between these two limits of a flat layer and of a rigid platelet occurs when the terms involving ε and c in (14) are of equal magnitude. The appropriate dimensionless parameter is $\alpha\mu_i/\mu_m$, where α is the aspect ratio. Taking both Poisson's ratio to be 1/4, the contributions to the two elements of the tensor in (14) are of equal magnitude when $\alpha\mu_i/\mu_m = 0.895$ and 0.764, respectively. In general then, we have

$$\alpha\frac{\mu_i}{\mu_m} = \begin{cases} \gg 1: & \text{Rigid platelets} \\ 0(1): & \text{Crossover} \\ \ll 1: & \text{Flat layers} \end{cases}$$

where, in the crossover regime, the platelets are neither "rigid" nor "flat".

The main feature of the present theory that distinguishes it from previous studies is the inclusion of the effects caused by the presence of the sharp corners or edges of the platelets. Although we have not calculated the matrix stresses around the corners, it is expected that they will be far from uniform. The same type of localized stress concentration is found near the edges of cracks, or more generally, voids with sharp vertices. For this reason we refer to the effects attributable to the presence of the sharp platelet corners as edge effects, even though the theory has assumed the platelets to be spheroids of smoothly varying curvature. It is clear from (14) that the moduli will always be less than those predicted by ignoring the edge effects ($\varepsilon = \infty$). Similarly, the moduli will always be less than the predictions based upon the approximation of the filler as a rigid material ($\mu_i \to \infty$).

4. RANDOMLY ORIENTED PLATELETS IN DILUTE CONCENTRATION

The effect of a dilute distribution of randomly oriented similar platelets can be determined by averaging the tensor on the right hand side of (14) over all orientations. Any fourth order tensor with the symmetries expressed by (1) reduces to the isotropic tensor $(\frac{1}{3}A_{iijj}, \frac{1}{5}A_{ijij}-\frac{1}{15}A_{iijj})$ when averaged over all possible orthogonal transformations. In particular, a transversely isotropic tensor of the form $(2\alpha, 0, 0, 0, 2\delta, 0)$ becomes the isotropic tensor $(\frac{4}{3}\alpha, \frac{2}{15}\alpha+\frac{4}{5}\delta)$, see Walpole (1981).

Application of these generalities to the specific case of eqn (14) implies the effective bulk and shear moduli are κ and μ, where

$$\kappa = \kappa_m + \frac{4}{9}\left[\frac{3}{\varepsilon 32}\frac{(3-4\nu_m)}{\mu_m(1-\nu_m)} + \frac{1}{c\mu_i}\left(\frac{1-\nu_i}{1+\nu_i}\right)\right]^{-1} \tag{18}$$

$$\mu = \mu_m + \frac{1}{15}\left[\frac{3(3-4\nu_m)}{\varepsilon 32\mu_m(1-\nu_m)} + \frac{1}{c\mu_i}\left(\frac{1-\nu_i}{1+\nu_i}\right)\right]^{-1} + \frac{2}{5}\left[\frac{3(7-8\nu_m)}{\varepsilon 64\mu_m(1-\nu_m)} + \frac{1}{c\mu_i}\right]^{-1}. \tag{19}$$

The same comments apply to these formulae as to the previous result for aligned platelets. In particular, we note the change in moduli caused by randomly oriented rigid platelets in dilute concentration, $\varepsilon \ll 1$,

$$\frac{\kappa}{\kappa_m} = 1+\varepsilon f(\nu_m)+0(\varepsilon^2) \tag{20a}$$

$$\frac{\mu}{\mu_m} = 1+\varepsilon g(\nu_m)+0(\varepsilon^2) \tag{20b}$$

where the functions f and g are

$$f(x) = \frac{64}{9}\frac{(1-x)(1-2x)}{(1+x)(3-4x)} \tag{21a}$$

$$g(x) = \frac{32}{45}\frac{(1-x)(43-56x)}{(3-4x)(7-8x)}. \tag{21b}$$

These forms will be used in Section 6 to develop finite concentration predictions for rigid disks.

5. RESULTS FOR FINITE CONCENTRATIONS

5.1. *Aligned platelets*

There are many ways to extend the dilute concentration results of Sections 3 to higher, finite values of the volume fraction c and the disk density parameter ε. We will consider the application of several alternative effective medium approximations in Section 6 for the particular case of rigid platelets. The same approximate techniques could be applied to the general case of finite rigidity, but we do not present the details. Rather, we will confine our attention to the estimation of lower bounds on the effective modulus tensor **L**.

Unlike the bounds of Hashin and Shtrikman (1962) which make no assumptions on the geometry of the phases, the present bounds contain some shape dependence. These generalizations of the Hashin–Shtrikman bounds were first proposed by Walpole (1966a, b), and may be deduced from the Hashin–Shtrikman variational principle, as shown by Willis (1977). We restrict our attention to the lower bounds of the moduli, which by definition are such that all physically attainable moduli at the same concentration of filler exceed the lower bounds by a positive semidefinite tensor. The general expression for the lower bounds follows from Willis (1977) as **L**, where

$$c(\mathbf{L}-\mathbf{L}_m)^{-1} = (\mathbf{L}_i-\mathbf{L}_m)^{-1}+(1-c)\mathbf{P} \tag{22}$$

and **P** is the tensor defined in (9), i.e., it is related to the Eshelby tensor for a single platelet in an infinite matrix. The moduli **L** of (22) are identical to those predicted by the Mori–Tanaka effective medium approximation, as discussed, for example, by Norris (1989). We also note that in the limit in which the platelets act as infinite layers, i.e., $\alpha \to 0$ in (9), then (22) provides the *exact* expression for the transversely isotropic moduli of the two-phase layered composite (Christensen, 1979a; Postma, 1955).

For the present applications, we can simplify (22) using the assumption that the inclusion is much stiffer than the matrix, to get

$$\mathbf{L} = \mathbf{L}_m+c[\mathbf{M}_i+(1-c)\mathbf{P}]^{-1}+0(\mathbf{L}_m). \tag{23}$$

Note that this expression ignores changes in the moduli of the order of the original matrix moduli. The **P** of (23) is the same as in (9), and therefore the inverse in (23) can be performed explicitly. Noting that $(1-c)/\varepsilon \sim 1/\varepsilon$, it is easy to see that (23) actually reduces to the same expression as in (14).

It may at first appear surprising that the lower bound estimate has the same form as the dilute result; however, this estimate agrees with the bound of Willis (1980, 1981), valid in the limit of rigid platelets. In the opposite limit of flat layers, the bound agrees with the exact results of Postma (1955) for media composed of stacks of isotropic layers, with the appropriate approximations made for the platelet being much stiffer than the matrix. The expression (14) is also restricted to values of c such that $1-c = 0(1)$, i.e., the limiting value of $c \to 1$ is not covered. Finally, we reiterate that (14) is correct only to order $\mathbf{L}_m$. The out of plane moduli may not be given accurately by (14), but the in-plane moduli, which change the most, will be.

5.2. *Randomly oriented platelets*

The appropriate generalization of the Walpole–Willis bound (22) is achieved by re-writing it as

$$c(\mathbf{L}-\mathbf{L}_m)^{-1} = [c\mathbf{I}+(1-c)\mathbf{T}^{-1}](\mathbf{L}_i-\mathbf{L}_m)^{-1} \tag{24}$$

and then replacing the strain concentration tensor **T** by its average over all orientations. However, according to (8)

$$\begin{aligned}(\mathbf{L}_i-\mathbf{L}_m)\mathbf{T} &= \mathbf{P}+\mathbf{L}_i-\mathbf{L}_m\\ &= [\mathbf{P}+\mathbf{M}_i][\mathbf{I}+0(\mathbf{L}_m\mathbf{M}_i)]\end{aligned} \tag{25}$$

and since $\mathbf{L}_m\mathbf{M}_i$ is small by assumption, we have

$$\mathbf{L} = \mathbf{L}_m+c[c(\mathbf{L}_i-\mathbf{L}_m)^{-1}+(1-c)\langle(\mathbf{P}+\mathbf{M}_i)^{-1}\rangle^{-1}]^{-1}+0(\mathbf{L}_m) \tag{26}$$

where the brackets $\langle\ \rangle$ signify the orientational average. The bounds on the bulk and shear moduli are then κ and μ, which follow from (18), (19) and (26), as

$$(\kappa-\kappa_m)^{-1} = (\kappa_i-\kappa_m)^{-1}+(1-c)\frac{9}{4}\left[\frac{1}{c\mu_i}\left(\frac{1-v_i}{1+v_i}\right)+\frac{3}{\varepsilon 32\mu_m}\left(\frac{3-4v_m}{1-v_m}\right)\right] \tag{27}$$

$$(\mu-\mu_m)^{-1} = (\mu_i-\mu_m)^{-1}+(1-c)\left\{\frac{1}{15}\left[\frac{1}{c\mu_i}\left(\frac{1-v_i}{1+v_i}\right)+\frac{3}{\varepsilon 32\mu_m}\left(\frac{3-4v_m}{1-v_m}\right)\right]^{-1} + \frac{2}{5}\left[\frac{1}{c\mu_i}+\frac{3}{\varepsilon 64\mu_m}\left(\frac{7-8v_m}{1-v_m}\right)\right]^{-1}\right\}^{-1}. \tag{28}$$

These expressions produce the correct values as $c \to 1$ and as $c \to 0$. In the limit in which the platelets act like flat layers, the terms involving ε disappear, and it is a straightforward matter of algebra to show that (27) and (28) coincide with the Hashin–Shtrikman (1962) upper bounds on the isotropic moduli of a two-phase composite made of (κ_m, μ_m) and (κ_i, μ_i). This is to be expected, since it is known (Norris, 1989) that the Mori–Tanaka theory for randomly oriented flat layers of stiffer material embedded in the more compliant material predicts the *H–S* upper bounds. However, in the present situation, the upper bounds have been shown to follow from an expression for the *lower bounds*. Hence the estimates must be exact for the flat layer limit. This limit is also equivalent to an isotropic "rank-n laminate," with $n \geq 3$. The theory of rank-n laminates is discussed by, for example, Francfort and Murat (1986). These classes of composites have interesting theoretical properties, not least of which is that they can be realized, in principle, by a well defined lamination process on successively smaller length scales.

In general, the moduli given by (27) and (28) are less than those predicted on the basis of a flat layer approximation for the platelets. The difference is due to the edge effects and is contained in the terms involving ε.

6. EFFECTIVE MEDIUM THEORIES FOR FINITE CONCENTRATIONS OF RANDOMLY ORIENTED RIGID DISKS

6.1. *Lower bounds*

In the limit that the platelets are infinitely rigid, the expressions (27) and (28) for the lower bound estimates of the moduli reduce to

$$\frac{\kappa}{\kappa_m} = 1+\varepsilon f(v_m) \tag{29a}$$

$$\frac{\mu}{\mu_m} = 1+\varepsilon g(v_m) \tag{29b}$$

where f and g are defined in (21a, b). As mentioned in Section 5, these results are the same as the extension of the dilute concentration estimates to finite values of ε. Note that as ε increases, the effective moduli increase monotonically, with no critical value for ε at which the medium becomes effectively rigid.

6.2. *A self-consistent theory*

A "self-consistent" estimate can be obtained by first writing the dilute result (20a, b) for randomly oriented rigid disks as

$$\mathbf{M} = \mathbf{M}_m[\mathbf{I}-\varepsilon\mathbf{B}(v_m)] \tag{30}$$

where

$$\mathbf{B}(v) = (f(v), g(v)). \tag{31}$$

The tensor $\mathbf{B}$ accounts for the effect of the matrix through its dependence upon v_m. The self-consistent method, as developed by Hill (1965) and Budiansky (1965), allows for interaction between inclusions at finite concentration by embedding the inclusion in the effective medium rather than in the matrix. It can be shown in the present context, that the self-consistent approximation amounts to making the substitution $\mathbf{B}(\nu_m) \to \mathbf{B}(\nu)$, where ν is the Poisson's ratio of the effective medium. This yields an implicit equation

$$\mathbf{M} = \mathbf{M}_m[\mathbf{I} - \varepsilon \mathbf{B}(\nu)] \tag{32}$$

which reduces to explicit expressions for the bulk and shear moduli κ, μ, as

$$\frac{\kappa_m}{\kappa} = 1 - \varepsilon f(\nu) \tag{33a}$$

$$\frac{\mu_m}{\mu} = 1 - \varepsilon g(\nu) \tag{33b}$$

and an implicit equation for the effective Poisson's ratio ν is obtained from the ratio $\kappa/\mu = 2(1+\nu)/[3(1-2\nu)]$; thus,

$$\left(\frac{1+\nu}{1-2\nu}\right)\left[\frac{1-\varepsilon f(\nu)}{1-\varepsilon g(\nu)}\right] = \frac{1+\nu_m}{1-2\nu_m}. \tag{34}$$

Unlike the estimates for the lower bounds, the effective moduli of (33a, b) and (34) predict a rigid effective medium for a finite value of ε. As ε approaches this critical value, ε_c, from below, both κ and μ tend to infinity, and $\nu \to \nu_c$ where ν_c is the unique root of

$$f(\nu_c) = g(\nu_c) \tag{35}$$

that lies in $(-1, \frac{1}{2})$. Simplifying (35) implies that

$$24\nu_c^2 - 23\nu_c + 3 = 0 \tag{36}$$

or

$$\nu_c = \frac{6}{23+\sqrt{241}} = 0.155746\ldots. \tag{37}$$

The critical concentration is then

$$\begin{aligned}\varepsilon_c &= 1/f(\nu_c)\\ &= \frac{9}{16} + \frac{27}{16(1+\sqrt{241})}\\ &= 0.6646\ldots.\end{aligned} \tag{38}$$

The self-consistent theory therefore predicts a rigidity threshold at ε_c. This phenomenon is similar to the threshold in Budiansky and O'Connell's (1976) theory of cracked solids. They found, using a similar self-consistent theory, that the bulk and shear moduli vanish when the crack density parameter, which is exactly the same as our disk density parameter ε, approached the value 9/16. The present threshold exceeds this value, as can be seen from (38). The other point in common with the theory of Budiansky and O'Connell is that a unique, limiting value is obtained for the Poisson's ratio as the threshold is approached. The value for cracks is zero, whereas for rigid disks we have ν_c as given by (37). One should not put undue emphasis on the exact value of the critical concentration; in particular it does not equal the percolation threshold for a random assortment of circular disks. However, the qualitative feature of the appearance of a rigidity threshold is physically appealing since one can expect the composite to eventually assume the stiffness of the platelets, if the platelets are allowed to become interpenetrating and connected. The single particle model upon which the self-consistent theory is based does not include such phenomena, and so one should not put too much credence in the quantitative predictions at high values of the concentration.

It should be noted that if the matrix is such that $v_m = v_c$, then (34) is satisfied by $v = v_c$ for all $\varepsilon < \varepsilon_c$. The Poisson's ratio of the effective medium therefore has a fixed point at v_c, at which value the moduli of (33a, b) become simply

$$\kappa = \frac{\kappa_m}{1-\varepsilon/\varepsilon_c}, \quad 0 \leqslant \varepsilon < \varepsilon_c \tag{39a}$$

$$\mu = \frac{\mu_m}{1-\varepsilon/\varepsilon_c}, \quad 0 \leqslant \varepsilon < \varepsilon_c. \tag{39b}$$

The value $v = v_c$ is also a fixed point for the lower bound estimates (29a, b), at which value

$$\kappa = \kappa_m\left(1+\frac{\varepsilon}{\varepsilon_c}\right), \quad 0 \leqslant \varepsilon < \infty \tag{40a}$$

$$\mu = \mu_m\left(1+\frac{\varepsilon}{\varepsilon_c}\right), \quad 0 \leqslant \varepsilon < \infty. \tag{40b}$$

6.3. *The differential scheme*

The dilute concentration result (30) can be expressed as

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}\varepsilon} = -\mathbf{M}_m\mathbf{B}(v_m), \quad \varepsilon = 0. \tag{41}$$

This can serve as the generator of a system of ordinary differential equations for $\mathbf{M} = \mathbf{M}(\varepsilon)$ by extending it to finite values of ε as

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}\varepsilon} = -\mathbf{M}(\varepsilon)\mathbf{B}(v(\varepsilon)), \quad \varepsilon \geqslant 0, \tag{42}$$

with initial condition $\mathbf{M}(0) = \mathbf{M}_m$. This type of differential effective medium theory is completely analogous to that for cracked solids, which is discussed at length by Hashin (1988). It should be noted that the same eqn (42) is obtained whether we start with the dilute change in compliance, as we have done here, or with the dilute form of the stiffness tensor.

It is not difficult to extract from (42) a single differential equation for the Poisson's ratio $v(\varepsilon)$,

$$\frac{\mathrm{d}v}{\mathrm{d}\varepsilon} = \frac{32}{13}\frac{(1-v)(1-2v)(24v^2-23v+3)}{(3-4v)(7-8v)} \tag{43}$$

with $v(0) = v_m$. This may be integrated by the method of partial fractions. Let v_e be the extraneous root of (36), i.e., $v_e = (8v_c)^{-1}$, or

$$v_e = 0.802587\ldots \tag{44}$$

then (43) integrates to

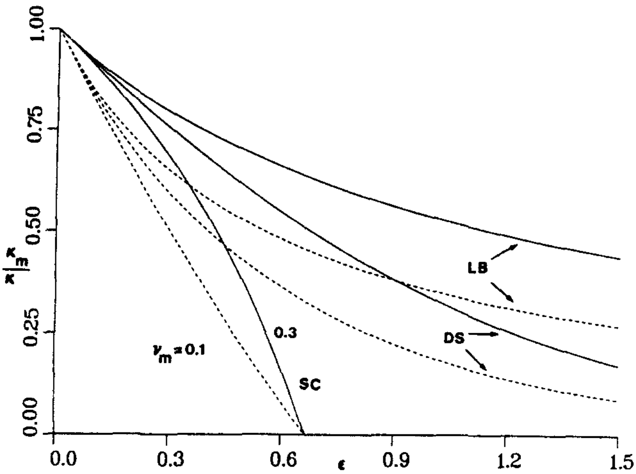

Fig. 1. The effective bulk modulus of a composite containing randomly oriented rigid disks, according to the lower bound estimates (LB), the self-consistent method (SC) and the differential scheme (DS): solid curves, $\nu_m = 0.3$; dashed curves, $\nu_m = 0.1$. Note that, since the reciprocal of κ is plotted, the lower bounds appear as the highest curves.

$$\varepsilon = \frac{5}{512}\left\{12\ln\left(\frac{\nu-1}{\nu_m-1}\right)+\frac{288}{5}\ln\left(\frac{\nu-0.5}{\nu_m-0.5}\right)+\frac{(3-4\nu_c)(7-8\nu_c)}{(\nu_c-1)(\nu_c-0.5)(\nu_c-\nu_e)}\ln\left(\frac{\nu-\nu_c}{\nu_m-\nu_c}\right)\right.$$
$$\left.+\frac{(3-4\nu_e)(7-8\nu_e)}{(\nu_e-1)(\nu_e-0.5)(\nu_e-\nu_c)}\ln\left(\frac{\nu-\nu_e}{\nu_m-\nu_e}\right)\right\}. \quad (45)$$

The following equation for κ as a function of ν can be obtained from (42) and (43),

$$\frac{d\ln\kappa}{d\nu} = \frac{10}{3}\frac{(7-8\nu)}{(1+\nu)(24\nu^2-23\nu+3)}. \quad (46)$$

Again using the method of partial fractions, with the appropriate initial condition, we get

$$\ln\left(\frac{\kappa}{\kappa_m}\right) = \ln\left(\frac{\nu+1}{\nu_m+1}\right)+\frac{(7-8\nu_c)(\nu_e+1)}{15(\nu_c-\nu_e)}\ln\left(\frac{\nu-\nu_c}{\nu_m-\nu_c}\right)+\frac{(7-8\nu_e)(\nu_c+1)}{15(\nu_e-\nu_c)}\ln\left(\frac{\nu-\nu_e}{\nu_m-\nu_e}\right). \quad (47)$$

The solution to the differential scheme then follows by first solving (45) for $\nu = \nu(\varepsilon)$, which is then substituted into (47) to give $\kappa = \kappa(\varepsilon)$, and the shear modulus can be calculated from $\mu = 3\kappa(0.5-\nu)/(1+\nu)$. These equations are similar in form to those obtained by Zimmerman (1985) in applying the differential scheme to cracked solids.

It is interesting to note that $\nu = \nu_c$ is again a fixed point, i.e., if $\nu_m = \nu_c$ then $\nu = \nu_c$ for all $\varepsilon > 0$, and the bulk and shear moduli become simply

$$\kappa = \kappa_m \exp(\varepsilon/\varepsilon_c) \quad (48a)$$

$$\mu = \mu_m \exp(\varepsilon/\varepsilon_c). \quad (48b)$$

6.4. *Numerical comparisons and discussion*

The three theories discussed above are compared in Figs 1, 2 and 3 for two values of the matrix Poisson's ratio: one less than and the other greater than the value ν_c, which is a fixed point for all three theories. Of the three, only the self-consistent theory predicts a finite rigidity threshold at ε_c. As expected, the self-consistent and differential schemes give moduli greater than those of the lower bounds. The general behavior of the curves in Figs

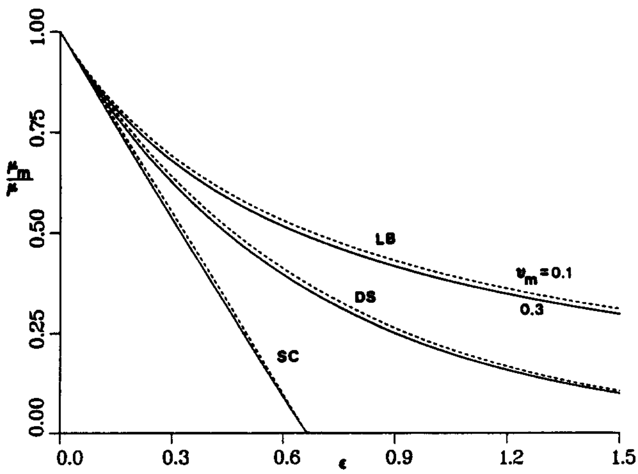

Fig. 2. The same as Fig. 1, but for the effective shear modulus.

1 and 2 is very reminiscent of similar curves for the effective moduli of solids containing cracks (Budiansky and O'Connell, 1976; Laws and Brockenbrough, 1987; Hashin, 1988). It can be seen from the relevant equations that $\nu \to \nu_c$ as $\varepsilon \to \varepsilon_c$ in the self-consistent theory and as $\varepsilon \to \infty$ in the differential scheme. However, the lower bounds estimates predict a limiting value of ν that depends upon ν_m as $\varepsilon \to \infty$, viz., $\nu \to (9-8\nu_m)/(61-72\nu_m)$.

Although we have a multiplicity of theories at our disposal, it is a simple fact that a given composite has only set of moduli, which begs the inevitable question: which theory is right? The simple answer is none of the above. However, there are arguments to be made in favor of each theory. First, the estimates (29a, b) provide us with confident lower bounds: any physical example must comply with these. The favorable features of the self-consistent theory have been discussed by Hill (1965), Budiansky (1965), and Budiansky and O'Connell (1976), among others. In the present context, we note that it displays the intuitively appealing feature of a finite rigidity threshold. However, as discussed in Section 6.2, there is no reason to expect that the value ε_c is actually correct. The differential scheme, like the self-consistent method, is essentially a mathematical technique. In contrast to both the lower bounds and the self-consistent method, the differential scheme has the significantly property that it is realizable. Thus, the predicted moduli can be realized, in principle, by successively embedding larger and larger disks into the matrix material. In practice, the differential scheme can be expected to be applicable to a composite containing a distribution

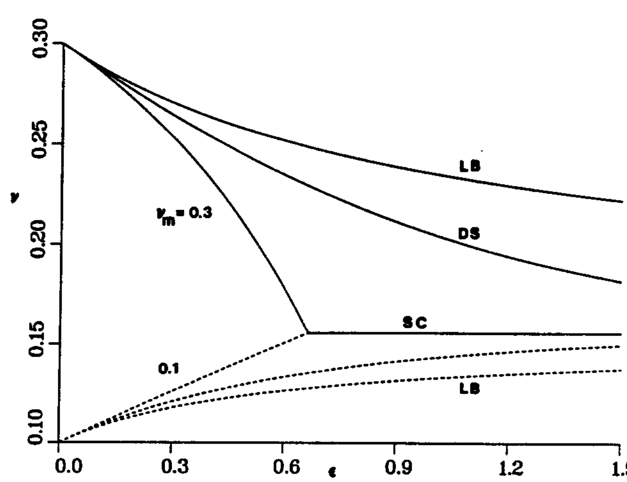

Fig. 3. The Poisson's ratio corresponding to Figs 1 and 2.

of disk sizes. The self-consistent scheme, as suggested by Budiansky and O'Connell (1976), may be better suited to cases in which only one disk size is present.

7. CONCLUSIONS

A theory for the effective moduli of platelet reinforced solids has been developed. The main innovation in the present theory as compared with previous studies is the inclusion of finite size or edge effects, which tend to diminish the effective moduli. The edge effects become critical when the reinforcing material becomes rigid. The theory of solids containing rigid disks has been explored in detail, and it has been shown to be very similar to the theory of solids permeated by circular cracks. The self-consistent theory predicts a finite rigidity threshold, and it has been shown that the Poisson's ratio of the composite will probably tend towards the value 0.1557... as the concentration of rigid disks is increased.

REFERENCES

Boucher, S. (1974). On the effective moduli of isotropic two-phase elastic composites. *J. Comp. Mater.* **8**, 82–89.

Budiansky, B. (1965). On the elastic moduli of some heterogeneous materials. *J. Mech. Phys. Solids* **13**, 223–227.

Budiansky, B. and O'Connell, R. J. (1976). Elastic moduli of a cracked solid. *Int. J. Solids Structures* **12**, 81–97.

Christensen, R. M. (1979a). Isotropic properties of platelet-reinforced media. *J. Eng. Mat. Tech.* **101**, 299–303.

Christensen, R. M. (1979b). *Mechanics of Composite Materials*. Wiley Interscience, New York.

Francfort, G. A. and Murat, F. (1986). Homogenization and optimal bounds in linear elasticity. *Arch. Rat. Mech. Anal.* **94**, 307–334.

Hashin, Z. (1988). The differential scheme and its application to cracked materials. *J. Mech. Phys. Solids* **36**, 719–734.

Hashin, Z. and Shtrikman, S. (1962). On some variational principles in anisotropic and nonhomogeneous elasticity. *J. Mech. Phys. Solids* **10**, 335–342.

Hill, R. (1965). A self-consistent mechanics of composite materials. *J. Mech. Phys. Solids* **13**, 213–222.

Laws, N. and Brockenbrough, J. R. (1987). The effect of micro-crack systems on the loss of stiffness of brittle solids. *Int. J. Solids Structures* **23**, 1247–1268.

Laws, N. and Dvorak, G. J. (1987). The effect of fiber breaks and aligned penny shaped cracks on the stiffness and energy release rates in unidirectional composites. *Int. J. Solids Structures* **23**, 1269–1283.

Maine, F. W. and Shepherd, P. D. (1974). Mica reinforced plastics: A review. *Composites* **5**, 193–200.

Mura, T. (1982). *Micromechanics of Defects in Solids*. Martinus Nijhoff, The Hague.

Norris, A. N. (1985). A differential scheme for the effective moduli of composites. *Mechanics of Materials* **4**, 1–16.

Norris, A. N. (1989). An examination of the Mori–Tanaka effective medium approximation for multiphase composites. *J. Appl. Mech.* **56**, 83–88.

Postma, G. W. (1955). Wave propagation in a stratified medium. *Geophysics* **20**, 780–806.

Walpole, L. J. (1966a). On bounds for the overall elastic moduli of inhomogeneous systems: I. *J. Mech. Phys. Solids* **14**, 151–162.

Walpole, L. J. (1966b). On bounds for the overall elastic moduli of inhomogeneous systems: II. *J. Mech. Phys. Solids* **14**, 289–301.

Walpole, L. J. (1969). On the overall elastic moduli of composite materials. *J. Mech. Phys. Solids* **17**, 235–251.

Walpole, L. J. (1981). Elastic behaviour of composite materials: Theoretical foundations. In *Advances in Applied Mechanics* (Edited by C. S. Yih), pp. 169–242. Academic Press, New York.

Willis, J. R. (1977). Bounds and self-consistent estimates for the overall properties of anisotropic composites. *J. Mech. Phys. Solids* **25**, 185–202.

Willis, J. R. (1980). A polarization approach to the scattering of elastic waves—II. Multiple scattering from inclusions. *J. Mech. Phys. Solids* **28**, 307–327.

Willis, J. R. (1981). Variational and related methods for the overall properties of composites. In *Advances in Applied Mechanics* (Edited by C. S. Yih), pp. 1–78. Academic Press, New York.

Wu, T. T. (1966). The effect of inclusion shapes on the elastic moduli of a two-phase material. *Int. J. Solids Structures* **2**, 1–8.

Zimmerman, R. W. (1985). The effect of microcracks on the elastic moduli of brittle materials. *J. Mat. Sci. Lett.* **4**, 1457–1460.